PH2101 Quantum Mechanics I
AY1415 Final Exam Suggested solutions

by some kind soul

May 27, 2016

1 MCQ

1. B   2. A   3. B   4. B   5. A   6. B   7. A   8. C   9. A   10. A

2 Q11

The probability density function (PDF) of velocity follows Maxwell-Boltzmann Distribution

$$f(v) = \sqrt{\left(\frac{m}{2\pi KT}\right)^3} 4\pi v^2 e^{\frac{-mv^2}{2KT}} \tag{2.1}$$

In this case, the most probable velocity may not give the most probable de Broglie wavelength, $\lambda = \frac{h}{p} = \frac{h}{mv}$ since the velocity follow a continuous random variable. Thus, we need to do transformation of random variable. Let $p(\lambda)$ be the probability density function (PDF) of λ

$$p(\lambda)|d\lambda| = f(v)|dv| \tag{2.2}$$

$$p(\lambda) = f(v)\frac{|dv|}{|d\lambda|} = f\left(\frac{h}{m\lambda}\right)\frac{h}{m\lambda^2}$$

where $v = \frac{h}{m\lambda}$, $\frac{dv}{d\lambda} = -\frac{h}{m\lambda^2}$. Let $c = 4\pi(\frac{m}{2\pi KT})^{\frac{3}{2}}$, $\alpha = \frac{m}{2KT}$, and $\beta = \frac{h}{m}$

$$\begin{aligned} p(\lambda) &= c\left(\frac{h}{m\lambda}\right)^2 e^{-\alpha(\frac{h}{m\lambda})^2}\frac{h}{m\lambda^2} \\ &= c\frac{h^3}{m^3\lambda^4}e^{-\alpha h^2/m^2\lambda^2} \\ &= \frac{c\beta^3}{\lambda^4}e^{-\alpha\beta^2/\lambda^2} \end{aligned} \tag{2.3}$$

In order to find λ that gives maximum $p(\lambda)$, we need to differentiate equation 2.3 and let it equal to 0.

$$\begin{aligned} \frac{d}{d\lambda}(p(\lambda)) &= c\beta^3\left\{e^{-\alpha\beta^2/\lambda^2}\left(\frac{-4}{\lambda^5}\right) + \left(\frac{1}{\lambda^4}\right)e^{-\alpha\beta^2/\lambda^2}\left(\frac{2\alpha\beta^2}{\lambda^3}\right)\right\} \\ &= c\beta^3\left(\frac{2}{\lambda^5}\right)e^{-\alpha\beta^2/\lambda^2}\left\{\frac{\alpha\beta^2}{\lambda^2} - 2\right\} = 0 \end{aligned} \tag{2.4}$$

Thus, we have the following as a result 2.4

$$\frac{\alpha\beta^2}{\lambda^2} = 2$$
$$\lambda = \beta\sqrt{\frac{\alpha}{2}} \tag{2.5}$$

Hence,

$$\lambda = \frac{h}{m}\sqrt{\frac{m}{4KT}} = \frac{h}{2\sqrt{mKT}}$$
$$\lambda = \frac{6.63 \cdot 10^{-34}}{2\sqrt{1.675 \cdot 10^{-27} 1.381 \cdot 10^{-23} 400}} = 1.09 \times 10^{-10}\,\mathrm{m} \tag{2.6}$$

3 Q12

1. Principal Number: n is positive integer, n $\in \mathbb{Z}^+$
2. Orbital Angular Momentum Number: $l \in [0, n-1]$
3. Magnetic Quantum Number: $m \in [-l, l]$
4. Degeneracy: $\sum_{l=0}^{n-1} 2l + 1 = n^2$

4 Q13

The nomenclature used in this question is as such:

1. the raising/creation operator:
$$\hat{a}^\dagger = \frac{m\omega\hat{x} - i\hat{p}}{\sqrt{2m\hbar\omega}} \tag{4.1}$$
2. the lowering/annihilation operator:
$$\hat{a} = \frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\hbar\omega}} \tag{4.2}$$
3. The position operator:
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^\dagger + \hat{a}) \tag{4.3}$$
4. the momentum operator:
$$\hat{p} = i\,\sqrt{\frac{\hbar\omega m}{2}}(\hat{a}^\dagger - \hat{a}) \tag{4.4}$$
5. the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, which acts as such on energy eigenstate $|\psi_n\rangle$:
$$\hat{a}^\dagger\hat{a}\;|\psi_n\rangle = \hat{N}\;|\psi_n\rangle = n\,|\psi_n\rangle \tag{4.5}$$

6. The commutator $[\hat{a}, \hat{a}^\dagger] = I$, where I is the identity operator.

We should aim to reduce this question to one in which we can exploit the simplicity of the actions of $[\hat{a}, \hat{a}^\dagger]$ and $\hat{N}$. Thus, as we proceed to construct the $\hat{p}^2$ and $\hat{x}^2$ operators, we should strive to express them in terms of the above operators. Glozing over trivial mathematical manipulation, you should obtain:

$$\begin{aligned} \hat{x}^2 &= \frac{\hbar}{2m\omega}(\hat{a}^{\dagger 2} + 2\hat{N} + I + \hat{a}^2) \\ \hat{p}^2 &= \frac{-\hbar m\omega}{2}(\hat{a}^{\dagger 2} - 2\hat{N} - I + \hat{a}^2)\ . \end{aligned} \tag{4.6}$$

We want to show the following:

- $\langle V \rangle = \langle \frac{1}{2} m\omega^2 x^2 \rangle = \frac{1}{2}\langle E \rangle$
- $\langle T \rangle = \langle \frac{p^2}{2m} \rangle = \frac{1}{2}\langle E \rangle$

where $\langle E \rangle = \hbar\omega(n + \frac{1}{2})$. Using the result mentioned above, we can do the following

$$\begin{aligned} \langle V \rangle &= \langle \psi_n | \frac{1}{2} m\omega^2 \hat{x}^2 | \psi_n \rangle \\ &= \frac{\hbar\omega}{4} \langle \psi_n | (\hat{a}^{\dagger 2} + 2\hat{N} + I + \hat{a}^2) | \psi_n \rangle \\ &= \frac{\hbar\omega}{4} \left[\langle \psi_n | a^{\dagger 2} | \psi_n \rangle + \langle \psi_n | 2\hat{N} | \psi_n \rangle + \langle \psi_n | I | \psi_n \rangle + \langle \psi_n | \hat{a}^2 | \psi_n \rangle \right] \\ &= \frac{\hbar\omega}{4} \left[\langle \psi_n | a^\dagger | \sqrt{n+1}\psi_{n+1} \rangle + \langle \psi_n | 2n\psi_n \rangle + \langle \psi_n | \psi_n \rangle + \langle \psi_n | \hat{a} | \sqrt{n}\psi_{n-1} \rangle \right] \\ &= \frac{\hbar\omega}{4} \left[\langle \psi_n | \sqrt{(n+1)(n+2)}\psi_{n+2} \rangle + 2n \langle \psi_n | \psi_n \rangle + \langle \psi_n | \psi_n \rangle + \langle \psi_n | \sqrt{n(n-1)}\psi_{n-2} \rangle \right] \\ &= \frac{\hbar\omega}{4} [2n+1] = \frac{1}{2}\hbar\omega[n + \frac{1}{2}] = \frac{1}{2}\langle E \rangle \end{aligned} \tag{4.7}$$

$$\begin{aligned} \langle T \rangle &= \langle \psi_n | \frac{\hat{p}^2}{2m} | \psi_n \rangle \\ &= \frac{-\hbar\omega}{4} \langle \psi_n | (\hat{a}^{\dagger 2} - 2\hat{N} - I + \hat{a}^2) | \psi_n \rangle \\ &= \frac{-\hbar\omega}{4} \left[\langle \psi_n | a^{\dagger 2} | \psi_n \rangle - \langle \psi_n | 2\hat{N} | \psi_n \rangle - \langle \psi_n | I | \psi_n \rangle + \langle \psi_n | \hat{a}^2 | \psi_n \rangle \right] \\ &= \frac{-\hbar\omega}{4} \left[\langle \psi_n | a^\dagger | \sqrt{n+1}\psi_{n+1} \rangle - \langle \psi_n | 2n\psi_n \rangle - \langle \psi_n | \psi_n \rangle + \langle \psi_n | \hat{a} | \sqrt{n}\psi_{n-1} \rangle \right] \\ &= \frac{-\hbar\omega}{4} \left[\langle \psi_n | \sqrt{(n+1)(n+2)}\psi_{n+2} \rangle - 2n \langle \psi_n | \psi_n \rangle - \langle \psi_n | \psi_n \rangle + \langle \psi_n | \sqrt{n(n-1)}\psi_{n-2} \rangle \right] \\ &= \frac{-\hbar\omega}{4} [-2n-1] = \frac{\hbar\omega}{4} [2n+1] = \frac{1}{2}\hbar\omega[n + \frac{1}{2}] = \frac{1}{2}\langle E \rangle \end{aligned} \tag{4.8}$$

$\dots\square$

5 Q14

(A) Eigenequation : $M\phi_m(x) = m\phi_m(x)$

(B) Let $\phi_m(x) = e^{2x}$. The corresponding result of operator M is $4e^{2x}$ with eigenvalue m is 4

(C) $\psi(x,0) = N(3\phi_1(x) + 4\phi_{100}(x))$
Normalization : $(3N)^2 + (4N)^2 = 1 \implies 25N^2 = 1 \implies N = \frac{1}{5}e^{i\theta}$
Hence, $\psi(x,0) = e^{i\theta}(\frac{3}{5}\phi_1(x) + \frac{4}{5}\phi_{100}(x))$

(D) Possible Measurement : $m = 1$ & $m = 100$
Probabilities : $\mathrm{P}(m=1) = \left|\frac{3}{5}e^{i\theta}\right|^2 = \frac{9}{25}$ & $\mathrm{P}(m=100) = \left|\frac{4}{5}e^{i\theta}\right|^2 = \frac{16}{25}$

(E) The time evolution can be found by expressing the wave function in terms of the energy eigenfunctions $u_n(x)$ first. Note that $\psi(x,0) = e^{i\theta}u_1(x)$ as given from question part (E). Therefore,

$$\psi(x,t) = \sum c_n u_n e^{\frac{-iE_n t}{\hbar}} = c_1 u_1 e^{\frac{-iE_1 t}{\hbar}} = e^{i\theta}(\frac{3}{5}\phi_1(x) + \frac{4}{5}\phi_{100}(x))e^{\frac{-iE_1 t}{\hbar}}$$

The last statement is obtained by expressing u_1 in term of ψ_1 and ψ_{100} according to information given from the question

(F) Possible Measurement at time t: $m = 1$ & $m = 100$
Probabilities at time t: $\mathrm{P}(m=1) = \left|\frac{3}{5}e^{i\theta}e^{\frac{-iE_1 t}{\hbar}}\right|^2 = \frac{9}{25}$ & $\mathrm{P}(m=100) = \left|\frac{4}{5}e^{i\theta}e^{\frac{-iE_1 t}{\hbar}}\right|^2 = \frac{16}{25}$

(G) Possible pair of $(m_0, m_t) = (1,1), (1,100), (100,1), (100,100)$
Initially, we need to express ϕ_1 and ϕ_{100} in term of u_1 and u_{100} to find the probability of observable m_t

$$\begin{pmatrix} u_1 \\ u_{100} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_{100} \end{pmatrix} \tag{5.1}$$

$$\begin{aligned} \implies \begin{pmatrix} \phi_1 \\ \phi_{100} \end{pmatrix} &= \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_{100} \end{pmatrix} \\ &= \frac{1}{-1} \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_{100} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} u_1 \\ u_{100} \end{pmatrix} \end{aligned}$$

$$\implies \begin{aligned} \phi_1 &= \frac{3}{5}u_1 + \frac{4}{5}u_{100} \\ \phi_{100} &= \frac{4}{5}u_1 - \frac{3}{5}u_{100} \end{aligned} \tag{5.2}$$

(a) Case 1: $m_0 = 1$ is measured
After the measured value of $m_0 = 1$, the wave function collapsed to $\boxed{\tilde{\psi}_1(x,0) = \phi_1(x)}$

Therefore,

$$\begin{aligned}
\tilde{\psi}_1(x,0) &= \frac{3}{5}u_1(x) + \frac{4}{5}u_{100}(x) \\
\implies \tilde{\psi}_1(x,t) &= \frac{3}{5}u_1(x)e^{\frac{-iE_1t}{\hbar}} + \frac{4}{5}u_{100}(x)e^{\frac{-iE_{100}t}{\hbar}} \\
&= \frac{3}{5}\left[\frac{3}{5}\phi_1 + \frac{4}{5}\phi_{100}\right]e^{\frac{-iE_1t}{\hbar}} + \frac{4}{5}\left[\frac{4}{5}\phi_1 - \frac{3}{5}\phi_{100}\right]e^{\frac{-iE_{100}t}{\hbar}} \\
&= \left[\frac{9}{25}e^{\frac{-iE_1t}{\hbar}} + \frac{16}{25}e^{\frac{-iE_{100}t}{\hbar}}\right]\phi_1(x) + \left[\frac{12}{25}e^{\frac{-iE_1t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100}t}{\hbar}}\right]\phi_{100}(x)
\end{aligned} \tag{5.3}$$

Hence,

$$\begin{aligned}
P(m_0 = 1|m_t = 1) &= \left|\frac{9}{25}e^{\frac{-iE_1t}{\hbar}} + \frac{16}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2 \\
P(m_0 = 1|m_t = 100) &= \left|\frac{12}{25}e^{\frac{-iE_1t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2
\end{aligned} \tag{5.4}$$

Thus,

$$\begin{aligned}
P(m_0 = 1, m_t = 1) &= P(m_0 = 1)P(m_0 = 1|m_t = 1) \\
&= \frac{9}{25}\left|\frac{9}{25}e^{\frac{-iE_1t}{\hbar}} + \frac{16}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2 \\
P(m_0 = 1, m_t = 100) &= P(m_0 = 1)P(m_0 = 1|m_t = 100) \\
&= \frac{9}{25}\left|\frac{12}{25}e^{\frac{-iE_1t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2
\end{aligned} \tag{5.5}$$

(b) Case 2 : $m_0 = 100$ is measured
After the measured value of $m_0 = 1$, the wave function collapsed to $\boxed{\tilde{\psi}_{100}(x,0) = \phi_{100}(x)}$
Therefore,

$$\begin{aligned}
\tilde{\psi}_{100}(x,0) &= \frac{4}{5}u_1(x) - \frac{3}{5}u_{100}(x) \\
\implies \tilde{\psi}_{100}(x,t) &= \frac{4}{5}u_1(x)e^{\frac{-iE_1t}{\hbar}} - \frac{3}{5}u_{100}(x)e^{\frac{-iE_{100}t}{\hbar}} \\
&= \frac{4}{5}\left[\frac{3}{5}\phi_1 + \frac{4}{5}\phi_{100}\right]e^{\frac{-iE_1t}{\hbar}} - \frac{3}{5}\left[\frac{4}{5}\phi_1 - \frac{3}{5}\phi_{100}\right]e^{\frac{-iE_{100}t}{\hbar}} \\
&= \left[\frac{12}{25}e^{\frac{-iE_1t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100}t}{\hbar}}\right]\phi_1(x) + \left[\frac{16}{25}e^{\frac{-iE_1t}{\hbar}} + \frac{9}{25}e^{\frac{-iE_{100}t}{\hbar}}\right]\phi_{100}(x)
\end{aligned} \tag{5.6}$$

Hence,

$$\begin{aligned}
P(m_0 = 100|m_t = 1) &= \left|\frac{12}{25}e^{\frac{-iE_1t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2 \\
P(m_0 = 100|m_t = 100) &= \left|\frac{16}{25}e^{\frac{-iE_1t}{\hbar}} + \frac{9}{25}e^{\frac{-iE_{100}t}{\hbar}}\right|^2
\end{aligned} \tag{5.7}$$

Thus,

$$\begin{aligned} P(m_0 = 100, m_t = 1) &= P(m_0 = 1)P(m_0 = 1|m_t = 1) \\ &= \frac{16}{25}\left|\frac{12}{25}e^{\frac{-iE_1 t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2 \\ P(m_0 = 100, m_t = 100) &= P(m_0 = 1)P(m_0 = 1|m_t = 100) \\ &= \frac{16}{25}\left|\frac{16}{25}e^{\frac{-iE_1 t}{\hbar}} + \frac{9}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2 \end{aligned} \tag{5.8}$$

In summary,

- $\mathrm{P}(m_0 = 1, m_t = 1) = \frac{9}{25}\left|\frac{9}{25}e^{\frac{-iE_1 t}{\hbar}} + \frac{16}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2$
- $\mathrm{P}(m_0 = 1, m_t = 100) = \frac{9}{25}\left|\frac{12}{25}e^{\frac{-iE_1 t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2$
- $\mathrm{P}(m_0 = 100, m_t = 1) = \frac{16}{25}\left|\frac{12}{25}e^{\frac{-iE_1 t}{\hbar}} - \frac{12}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2$
- $\mathrm{P}(m_0 = 100, m_t = 100) = \frac{16}{25}\left|\frac{16}{25}e^{\frac{-iE_1 t}{\hbar}} + \frac{9}{25}e^{\frac{-iE_{100} t}{\hbar}}\right|^2$

Several things to note,

i) All probability are time dependent

ii) When M is only measured at time t, the probability is time independent

iii) m_0 and m_t are DEPENDENT event. The probability measurement for m_t depends on what has been measured for m_0 as well as when it is measured

END OF SOLUTION