NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2014-2015

PH2104 - Analytic Mechanics

Nov/Dec 2014 | Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.

2. Answer **ALL** questions.

3. All questions carry **EQUAL** weight.

4. Answer each question beginning on a **FRESH** page of the answer book.

5. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring the **COURSE LECTURE NOTES** for reference.

6. Calculators may be used.

Question 1 (Coupled Oscillators)

(a) Explain what is meant by a *normal mode* for an oscillating system.

(3 marks)

(b) A model for a water molecule consists of two masses m, each connected to a central mass M by a spring. The two springs are identical and each has spring constant k. Consider the motion of this system when all the masses are constrained to lie along the same straight line, as shown below in Figure **1**.

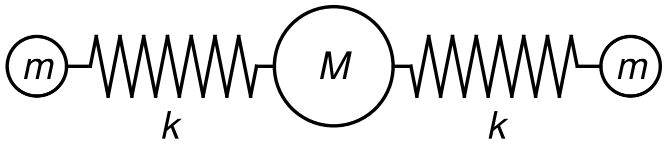

Figure 1: Three coupled oscillators.

Show that the normal mode (angular) frequencies are

$$0, \qquad (k/m)^{1/2}, \quad \text{and} \quad (k/m + 2k/M)^{1/2}$$

(13 marks)

(c) Find the displacements of the masses in each of the three normal modes.

(9 marks)

Question 2 (Rigid Body Rotation)

(a) Show that the moment of inertia of a uniform solid sphere of mass m and radius a about an axis through the centre:

$$\frac{2}{5}ma^2$$

(13 marks)

(b) The sphere rotates freely about a fixed vertical axis, passing through the centre. A small spider of mass $2m/5$ starts from the North pole and walks down the sphere. Let θ be the angle between the vertical and the radius from the sphere centre to the spider. If the sphere is initially rotating with angular velocity ω_0, show that the angular velocity when the spider is at θ is given by:

$$\omega(\theta) = \frac{\omega_0}{1 + \sin^2\theta}$$

(7 marks)

(c) The spider walks South with a constant speed, reaching the South pole after a time T. Find the angle that the sphere turned during this time.

(5 marks)

You may find the following integral useful:

$$\int_0^\pi \frac{d\theta}{1 + \sin^2\theta} = \frac{\pi}{\sqrt{2}}$$

Question 3 (Planetary Motion)

(a) A planet orbits a star under the influence of the star's gravitational attraction. State what quantity or quantities are conserved, and explain why the orbit lies in a plane.

(7 marks)

(b) The planet moves in an elliptical orbit. Prove that the maximum speed v_p occurs at the minimum radius r_p, and that the minimum speed v_a occurs at the maximum radius r_a.

(5 marks)

(c) Show that:

$$v_p^2 = \frac{2GMr_a}{(r_a + r_p)\, r_p}$$

where G is Newton's constant of universal gravitation and M is the mass of the star.

(8 marks)

(d) By writing down the analogous formula for v_a^2, show that that the product of the minimum and maximum speeds in the orbit is:

$$v_p v_a = \frac{GM}{a}$$

where a is the semi-major axis of the elliptical orbit.

(5 marks)

Question 4 (Lagrangian Dynamics)

A snowboarder with mass m travels on a smooth horizontal surface with velocity v, and approaches a wedge shaped object, characterised by an inclination angle α and mass M. The wedge is initially stationary, but free to move on the horizontal surface, as shown in Figure **2**.

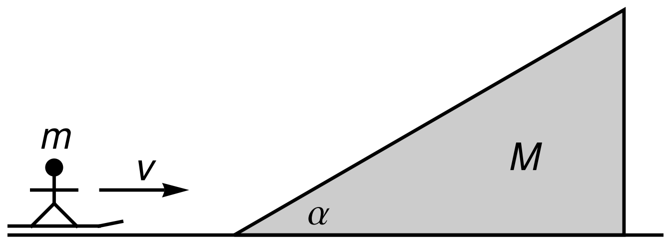

Figure 2: A snowboarder approaching a wedge shaped object.

(a) Neglecting friction, choose two suitable coordinates and write the total kinetic energy, potential energy and Lagrangian to describe the situation when the snowboarder is on the wedge.

(6 marks)

(b) Calculate the generalised forces and generalised momenta from the Lagrangian. Comment on any conserved quantities.

(6 marks)

(c) Calculate the deceleration of the snowboarder and the acceleration of the wedge.

(7 marks)

(d) The snowboarder has just enough momentum to reach the top of the wedge, without falling off. Write the velocity of the wedge (and snowboarder) when the snowboarder reaches this position in terms of m, M and v (in an observer's frame of reference).

(6 marks)

— End of Paper —

Question 1 Solution

(a)

Marks for any of the following:

- A normal mode is a pattern of motion in which all parts of a system oscillate with the same *frequency* and have a definite *phase* relation.
- The frequencies are resonant or *natural* frequencies of a system.
- The most general motion of a system is a *superposition* of its normal modes.
- The amplitudes of normal modes are in fixed ratios, while the overall normalization is *arbitrary.*
- Mathematically:

$$\vec{x}(t) = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_n \end{pmatrix} e^{i\omega t}$$

(b)

First write the forces on the three masses, as a function of their displacement:

$$\begin{aligned} F_1 &= k\,(x_2 - x_1) \\ F_2 &= k\,(x_3 - x_2) + k\,(x_1 - x_2) \\ F_3 &= k\,(x_2 - x_3) \end{aligned}$$

According to Newton's second law:

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let us now substitute for a trial normal mode solution, $\vec{x}(t) = \vec{A}e^{i\omega t}$.

$$-\omega^2 \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Rearranging:

$$\begin{pmatrix} -\frac{k}{m} + \omega^2 & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} + \omega^2 & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

The determinant of the above matrix must vanish for a solution, giving:

$$\left(-\frac{k}{m}+\omega^2\right)\left(\left(-\frac{2k}{M}+\omega^2\right)\left(-\frac{k}{m}+\omega^2\right)-\frac{k^2}{mM}\right)-\frac{k^2}{mM}\left(-\frac{k}{m}+\omega^2\right)=0$$

$$\left(-\frac{k}{m}+\omega^2\right)\left(-\frac{2k}{M}-\frac{k}{m}+\omega^2\right)\omega^2=0$$

Equality is satisfied for ω equal to one of the normal mode frequencies:

$$0, \qquad (k/m)^{1/2}, \qquad (k/m+2k/M)^{1/2}$$

(c)

The displacements are found by substituting back into the eigenvalue equation.

For $\omega = 0$:

$$0\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

This gives $A_1 = A_2$, $A_1 + A_3 = 2A_2$ and $A_2 = A_3$, corresponding to the normalized eigenvector, where all masses oscillate in-phase:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\omega^2 = k/m$:

$$-\frac{k}{m}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

This gives $A_2 = 0$, $\frac{A_1}{M} + \left(\frac{1}{m} - \frac{2}{M}\right)A_2 + \frac{A_3}{M} = 0$ and $2A_2 = 0$, corresponding to the normalized eigenvector, where the two outer masses oscillate out-of phase with the central mass stationary:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\omega^2 = k/m + 2k/M$:

$$-\left(\frac{k}{m}+\frac{2k}{M}\right)\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{M} & -\frac{2k}{M} & \frac{k}{M} \\ 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

This gives $-\frac{2A_1}{M} = \frac{A_2}{m}$, $-\frac{A_2}{m} = \frac{A_1+A_3}{M}$ and $-\frac{2A_3}{M}\frac{A_2}{m}$, corresponding to the normalized eigenvector:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \frac{1}{\sqrt{2}\sqrt{M^2+2m^2}}\begin{pmatrix} M \\ -2m \\ M \end{pmatrix}$$

Here the two outer masses oscillate in phase with one another, but out-of-phase with the central mass. If $M \gg m$, then the amplitude of the central mass is relatively small.

Question 2 Solution

(a)

From symmetry, the moment of inertia is the same about any axis.

The moment of inertia about the z-axis is:

$$
\begin{aligned}
I &= \rho \int (x^2 + y^2) dxdydz \\
&= \rho \int_0^a \int_0^\pi \int_0^{2\pi} \left(r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi\right) r^2 \sin\theta dr d\theta d\phi \\
&= 2\pi\rho \int_0^a \int_0^\pi r^4 \sin^3 dr d\theta \\
&= \frac{2\pi}{5}\rho a^5 \int_0^\pi \sin^3 dr d\theta
\end{aligned}
$$

The integral of $\sin^3\theta$ can be made using the substitution $u = \cos^{3/2}\theta$:

$$
\begin{aligned}
\int_0^\pi \sin^3\theta d\theta &= \int_0^\pi (1 - \cos^2\theta)\sin\theta d\theta \\
&= \int_0^\pi \sin\theta d\theta - \int_0^\pi \sin\theta\cos\theta d\theta \\
&= [-\cos\theta]_0^\pi + \frac{2}{3}\int_1^0 u du \\
&= 2 + \frac{1}{3}\left[u^2\right]_1^0 \\
&= \frac{4}{3}
\end{aligned}
$$

The density $\rho = m/V$, where the volume is:

$$
\begin{aligned}
V &= \int_0^a \int_0^\pi \int_0^{2\pi} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{2\pi a^3}{3}\int_0^\pi \sin\theta d\theta \\
&= \frac{2\pi a^3}{3}[-\cos\theta]_0^\pi = \frac{4\pi a^3}{3}
\end{aligned}
$$

Consequently, the moment of inertia is:

$$
I = \frac{2\pi}{5}\rho a^5 \frac{4}{3}\frac{3m}{4\pi a^3} = \frac{2}{5}ma^2
$$

(b)

The moment of inertial of the sphere and the spider about the vertical axis is:

$$
I_z = \frac{2}{5}ma^2 + \frac{2}{5}ma^2\sin^2\theta = \frac{2}{5}ma^2(1 + \sin^2\theta)
$$

From angular momentum conservation:

$$L = I(0)\omega_0 = I(\theta)\omega(\theta)$$

Therefore:

$$\omega(\theta) = \frac{I(0)}{I(\theta)}\omega_0 = \frac{\omega_0}{1+\sin^2\theta}$$

(c)

The position of the spider is given by $\theta = \frac{\pi t}{T}$.

The angle by which the sphere has rotated is:

$$\begin{aligned}\int_0^T \omega(t)dt &= \omega_0 \int_0^T \frac{1}{1+\sin^2\theta}dt \\ &= \frac{T}{\pi}\omega_0 \int_0^\pi \frac{1}{1+\sin^2\theta}d\theta \\ &= \frac{\omega_0 T}{\sqrt{2}}\end{aligned}$$

where we used the definite integral given in the question.

Question 3 Solution

(a)

Angular momentum and energy are conserved.

The angular momentum is $\vec{L} = \vec{r} \times \vec{p}$.

Since angular momentum is conserved, $\vec{r}$ and $\vec{p}$ must always lie in the plane orthogonal to $\vec{L}$.

(b)

The total energy in the system is:

$$\frac{1}{2}mv^2 - \frac{GmM}{r} = \text{constant}$$

Therefore, v is a maximum when r is a minimum; v is a minimum when r is a maximum.

(c)

The angular momentum is the same at r_a and r_p. Therefore:

$$\begin{aligned} mr_a v_a &= mr_p v_p \\ v_a &= \frac{r_p v_p}{r_a} \end{aligned}$$

The total energy must also be the same at r_a and r_p:

$$\begin{aligned} \frac{1}{2}mv_p^2 - \frac{GmM}{r_p} &= \frac{1}{2}mv_a^2 - \frac{GmM}{r_a} \\ \frac{1}{2}\left(v_p^2 - v_a^2\right) &= GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right) \\ \frac{v_p^2}{2}\left(1 - \frac{r_p^2}{r_a^2}\right) &= GM\left(\frac{r_a - r_p}{r_a r_p}\right) \\ v_p^2\left(r_a^2 - r_p^2\right) &= 2GM\left(r_a - r_p\right)\frac{r_a}{r_p} \\ v_p^2 &= \frac{2GMr_a}{\left(r_a + r_p\right) r_p} \end{aligned}$$

(d)

The analogous formula is:

$$v_a^2 = \frac{2GMr_p}{\left(r_a + r_p\right) r_a}$$

So:

$$\begin{aligned} v_a^2 v_p^2 &= \left(2GM\right)^2 \frac{r_a}{\left(r_a + r_p\right) r_p} \frac{r_p}{\left(r_a + r_p\right) r_a} \\ &= \frac{\left(2GM\right)^2}{\left(r_a + r_p\right)^2} \end{aligned}$$

Taking the square root:

$$v_a v_p = \frac{2GM}{r_a + r_p} = \frac{GM}{a}$$

since $r_a + r_p = 2a$

Question 4 Solution

(a)

Let us take X as the position of the left-hand corner of the wedge. Let us measure the snowboarder position x relative to the left-hand corner of the wedge. Then $y = x\tan\alpha$ and $\dot{y} = \dot{x}\tan\alpha$.

The kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}m\left(\left(\dot{x}+\dot{X}\right)^2+\dot{y}^2\right)+\frac{1}{2}M\dot{X}^2 \\ &= \frac{1}{2}m\left(1+\tan^2\alpha\right)\dot{x}^2+m\dot{x}\dot{X}+\frac{1}{2}\left(m+M\right)\dot{X}^2 \end{aligned}$$

The potential energy is:

$$U = mgy = mgx\tan\alpha$$

We can now write the Lagrangian:

$$L = T - U = \frac{1}{2}m\left(1+\tan^2\alpha\right)\dot{x}^2+m\dot{x}\dot{X}+\frac{1}{2}\left(m+M\right)\dot{X}^2 - mgx\tan\alpha$$

(b)

The generalized momenta are:

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x}\left(1+\tan^2\alpha\right)+m\dot{X} \\ p_X &= \frac{\partial L}{\partial \dot{X}} = m\dot{x}+\left(m+M\right)\dot{X} \end{aligned}$$

The generalized forces are:

$$\begin{aligned} F_x &= \frac{\partial L}{x} = -mg\tan\alpha \\ F_X &= \frac{\partial L}{X} = 0 \end{aligned}$$

Since the Lagrangian does not depend explicitly on X, there is a conserved momentum, p_X.

Since the Lagrangian does not depend explicitly on time, the Hamiltonian/total energy is conserved.

(c)

Since p_X is conserved:

$$p_X = m\dot{x}+\left(m+M\right)\dot{X} = \text{constant} = mv$$

Rearranging:

$$\begin{aligned} \dot{X} &= -\frac{m}{m+M}\dot{x} \\ \ddot{X} &= -\frac{m}{m+M}\ddot{x} \end{aligned}$$

Applying the Euler-Lagrange equation for the x coordinate:

$$\begin{aligned}
\dot{p_x} &= F_x \\
m\ddot{x}\left(1+\tan^2\alpha\right) + m\ddot{X} &= -mg\tan\alpha \\
\ddot{x}\left(1+\tan^2\alpha\right) - \frac{m}{m+M}\ddot{x} &= -g\tan\alpha \\
\ddot{x}\left(\frac{1}{\cos^2\alpha} - \frac{m}{m+M}\right) &= -g\frac{\sin\alpha}{\cos\alpha} \\
\ddot{x}\left(1 - \frac{m\cos^2\alpha}{m+M}\right) &= -g\sin\cos\alpha
\end{aligned}$$

Rearranging for $\ddot{x}$:

$$\begin{aligned}
\ddot{x} &= -\frac{g\sin\alpha\cos\alpha}{1-\frac{m\cos\alpha}{m+M}} \\
&= -\frac{(m+M)\,g\sin\alpha\cos\alpha}{M+m\sin^2\alpha}
\end{aligned}$$

Substituting back into the equation for $\ddot{X}$:

$$\ddot{X} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$$

Note that both $\ddot{x}$ and $\ddot{X}$ are constant!

(d)

The component of the snowboarder velocity in x is:

$$\dot{x} = v + \ddot{x}t$$

(where $\ddot{x}$ is constant. When the snowboarder reaches the end of the wedge at time t_f, it is assumed that $\dot{x} = 0$:

$$t_f = -\frac{v}{\ddot{x}}$$

The velocity of the wedge is:

$$\dot{X} = \ddot{X}t$$

where $\ddot{X}$ is constant and we assume that X is initially zero. Substituting t for t_f:

$$\dot{X} = -v\frac{\ddot{X}}{\ddot{x}} = \frac{mv}{m+M}$$