PH2101 Quantum Mechanics I
AY1516 Final Exam Suggested solutions

by some kind soul

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1 Q1

The wavelength of a photon, λ is 5500Å or $5.5 \cdot 10^{-7}$ m. The corresponding wavelength has an energy calculated as following

$$E_{photon} = \frac{hc}{\lambda} \tag{1.1}$$

The energy from each photon is $3.614 \cdot 10^{-19}$ Joule. The power level corresponds for 100 photons absorbed per second is $3.614 \cdot 10^{-17}$ Watt

2 Q2

The discrete wave function has the following value for each position

$$\Psi(x) = \begin{cases} 1 & 0 \leq x \leq 1, 4 \leq x \leq 6 \\ 2 & 2 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

The absolute square of the wave function is

$$|\Psi(x)|^2 = \begin{cases} 1 & 0 \leq x \leq 1, 4 \leq x \leq 6 \\ 4 & 2 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

However, the absolute square value of this wave function does not directly correspond to the probabilities of particle at certain position since the wave function is not normalised. We need to normalize the wave function by multiplying the wave function (or the absolute square of wave function) with normalization constant i.e make the area under the absolute square of wave function equal to 1. As the area under non-normalised absolute square of wave function is 7, the normalised absolute square is multiplied with constant $\frac{1}{7}$ which result in the following change

$$|\Psi_{norm}(x)|^2 = \begin{cases} \frac{1}{7} & 0 \leq x \leq 1, 4 \leq x \leq 6 \\ \frac{4}{7} & 2 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

Therefore, the probability that the particle will be found between $x = 4$ and $x = 6$ is the area under absolute square of normalized wave function which is $\frac{2}{7}$

3 Q3

We want to show the following:

$$\frac{d}{dt}\langle x^2\rangle = \frac{1}{m}\ \{\langle\hat{x}\hat{p}\rangle + \langle\hat{p}\hat{x}\rangle\}\ . \tag{3.1}$$

We start by using the generalized Ehrenfest Theorem, which states that for some observable A corresponding to the operator $\hat{A}$:

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle\ . \tag{3.2}$$

Hence, the relevant equation here takes the form:

$$\frac{d}{dt}\langle\hat{x}^2\rangle = \frac{1}{i\hbar}\langle[\hat{x}^2,\hat{H}]\rangle + \left\langle\frac{\partial(x^2)}{\partial t}\right\rangle\ . \tag{3.3}$$

We first proceed to evaluate the commutator $[\hat{x}^2,\hat{H}]$ acting on some general wavefunction ψ:

$$\begin{aligned}
[\hat{x}^2,\hat{H}]\ \psi &= \hat{x}^2\hat{H}\psi - \hat{H}\hat{x}^2\psi \\
&= x^2\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right) - \left[\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\psi x^2\right] \\
&= \frac{-\hbar^2 x^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi x^2 - \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi x^2] + V\psi x^2\right) \\
&= \frac{-\hbar^2 x^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{2m}\left[2\psi + 2x\frac{\partial\psi}{\partial x} + 2x\frac{\partial\psi}{\partial x} + x^2\frac{\partial^2\psi}{\partial x^2}\right] \\
&= \frac{-\hbar^2 x^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{m}\psi + \frac{2x\hbar^2}{m}\frac{\partial\psi}{\partial x} + \frac{\hbar^2 x^2}{2m}\frac{\partial^2\psi}{\partial x^2} \\
&= \frac{\hbar^2}{m}\psi + \frac{2x\hbar^2}{m}\frac{\partial\psi}{\partial x} \\
&= \left[\frac{\hbar^2}{m} + \frac{2x\hbar^2}{m}\frac{\partial}{\partial x}\right]\psi\ .
\end{aligned} \tag{3.4}$$

Having determined the commutator $[\hat{x}^2,\hat{H}]$, we can proceed to find the first term of equation 3.3:

$$\begin{aligned}
\frac{1}{i\hbar}\langle[\hat{x}^2,\hat{H}]\rangle &= \frac{1}{i\hbar}\int\psi^*[\hat{x}^2,\hat{H}]\psi\ dx \\
&= \frac{1}{i\hbar}\int\psi^*\left[\frac{\hbar^2}{m} + \frac{2x\hbar^2}{m}\frac{\partial}{\partial x}\right]\psi\ dx \\
&= \frac{1}{i\hbar}\left\{\frac{1}{m}\int\psi^*\hbar^2\psi\ dx + \frac{2}{m}\int\psi^* x\hbar^2\frac{\partial}{\partial x}\psi\ dx\right\} \\
&= \frac{1}{m}\left\{\int\psi^*(-i\hbar\psi)\ dx + 2\int\psi^*[x(-i\hbar\frac{\partial}{\partial x})\psi]\ dx\right\} \\
&= \frac{1}{m}\left\{\int\psi^*(-i\hbar\psi)\ dx + 2\int\psi^*(\hat{x}\hat{p}_x\psi)\ dx\right\}\ .
\end{aligned} \tag{3.5}$$

It would be wise at this point to invoke the canonical commutator:

$$[\hat{x}, \hat{p}_x] = i\hbar \implies [\hat{p}_x, \hat{x}] = -i\hbar \ , \tag{3.6}$$

which we can use to replace the term $-i\hbar\psi$. This yields:

$$\begin{aligned}
\frac{1}{i\hbar}\langle[\hat{x}^2, \hat{H}]\rangle &= \frac{1}{m}\left\{\int \psi^*[\hat{p}_x, \hat{x}] \ \psi \ dx + 2\int \psi^*(\hat{x}\hat{p}_x \ \psi) \ dx\right\} \\
&= \frac{1}{m}\left\{\int \psi^*[\hat{p}_x\hat{x} \ \psi - \hat{x}\hat{p}_x \ \psi] \ dx + 2\int \psi^*(\hat{x}\hat{p}_x \ \psi) \ dx\right\} \\
&= \frac{1}{m}\left\{\int \psi^*\hat{p}_x\hat{x} \ \psi \ dx - \int \psi^*\hat{x}\hat{p}_x \ \psi \ dx + 2\int \psi^*\hat{x}\hat{p}_x \ \psi \ dx\right\} \\
&= \frac{1}{m}\left\{\int \psi^*\hat{p}_x\hat{x} \ \psi \ dx + \int \psi^*\hat{x}\hat{p}_x \ \psi \ dx\right\} \\
&= \frac{1}{m}\{\langle\hat{p}_x\hat{x}\rangle + \langle\hat{x}\hat{p}_x\rangle\} \ .
\end{aligned} \tag{3.7}$$

Since x and t and independent of each other:

$$\left\langle\frac{\partial(x^2)}{\partial t}\right\rangle = 0 \ . \tag{3.8}$$

Hence, we have proven that:

$$\frac{d}{dt}\langle x^2\rangle = \frac{1}{m} \ \{\langle\hat{x}\hat{p}\rangle + \langle\hat{p}\hat{x}\rangle\} \tag{3.9}$$

. . . □

4 Q4

Show that the ground state of the quantum harmonic oscillator, $|\psi_0\rangle$, satisfies the Heisenberg uncertainty principle. The nomenclature used in this question is as such:

1. the raising/creation operator:

$$\hat{a}^\dagger = \frac{m\omega\hat{x} - i\hat{p}_x}{\sqrt{2m\hbar\omega}} \tag{4.1}$$

2. the lowering/annihilation operator:

$$\hat{a} = \frac{m\omega\hat{x} + i\hat{p}_x}{\sqrt{2m\hbar\omega}} \tag{4.2}$$

3. The position operator:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \ (\hat{a}^\dagger + \hat{a}) \tag{4.3}$$

4. the momentum operator:

$$\hat{p}_x = i\sqrt{\frac{\hbar\omega m}{2}}(\hat{a}^\dagger - \hat{a}) \tag{4.4}$$

5. the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$, which acts as such on energy eigenstate $|\psi_n\rangle$:

$$\hat{a}^\dagger \hat{a} \ |\psi_n\rangle = \hat{N} \ |\psi_n\rangle = n \, |\psi_n\rangle \tag{4.5}$$

6. The commutator $[\hat{a}, \hat{a}^\dagger] = I$, where I is the identity operator.

We should aim to reduce this question to one in which we can exploit the simplicity of the actions of $[\hat{a}, \hat{a}^\dagger]$ and $\hat{N}$. Thus, as we proceed to construct the $\hat{p}_x^2$ and $\hat{x}^2$ operators, we should strive to express them in terms of the above operators. Glozing over trivial mathematical manipulation, you should obtain:

$$\begin{aligned} \hat{x}^2 &= \frac{\hbar}{2m\omega}(\hat{a}^{\dagger 2} + 2\hat{N} + I + \hat{a}^2) \\ \hat{p}_x^2 &= \frac{-\hbar m\omega}{2}(\hat{a}^{\dagger 2} - 2\hat{N} - I + \hat{a}^2) \ . \end{aligned} \tag{4.6}$$

Now, we evaluate the expectation values of x, x^2, p_x, and p_x^2:

$$\begin{aligned} \langle x \rangle &= \langle \psi_0 | \, \hat{x} \, | \psi_0 \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} \, \langle \psi_0 | \, (\hat{a}^\dagger + \hat{a}) \, | \psi_0 \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} \, [\langle \psi_0 | \, \hat{a}^\dagger \, | \psi_0 \rangle + \langle \psi_0 | \, \hat{a} \, | \psi_0 \rangle] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \, [\langle \hat{a}\psi_0 | \psi_0 \rangle + \langle \psi_0 | \hat{a}\psi_0 \rangle] \end{aligned} \tag{4.7}$$

We know that $\hat{a} \, |\psi_0\rangle = 0$ (*i.e* the ground state is the lowest state). By orthogonality of the solutions, we have $\langle \psi_m | \psi_n \rangle = \delta_{mn}$. Hence, without evaluating the above, we know that both inner product terms vanish, leaving us with:

$$\boxed{\langle x \rangle = 0} \tag{4.8}$$

We encounter a similar case for $\langle \hat{p}_x \rangle$:

$$\begin{aligned} \langle p \rangle &= \langle \psi_0 | \, \hat{p}_x \, | \psi_0 \rangle \\ &= i \sqrt{\frac{\hbar m\omega}{2}} \, \langle \psi_0 | \, (\hat{a}^\dagger - \hat{a}) \, | \psi_0 \rangle \\ &= i \sqrt{\frac{\hbar m\omega}{2}} \, [\langle \psi_0 | \, \hat{a}^\dagger \, | \psi_0 \rangle - \langle \psi_0 | \, \hat{a} \, | \psi_0 \rangle] \\ &= i \sqrt{\frac{\hbar m\omega}{2}} \, [\langle \hat{a}\psi_0 | \psi_0 \rangle - \langle \psi_0 | \hat{a}\psi_0 \rangle] \\ \implies & \boxed{\langle p_x \rangle = 0} \end{aligned} \tag{4.9}$$

We now move on to find the expectation values of the squares of position and momentum:

$$\begin{aligned}\langle x^2\rangle &= \langle\psi_0|\,\hat{x}^2\,|\psi_0\rangle\\ &= \frac{\hbar}{2m\omega}\,\langle\psi_0|\,(\hat{a}^{\dagger 2}+2\hat{N}+I+\hat{a}^2)\,|\psi_0\rangle\\ &= \frac{\hbar}{2m\omega}[\langle\psi_0|\,\hat{a}^{\dagger 2}\,|\psi_0\rangle + 2\,\langle\psi_0|\,\hat{N}\,|\psi_0\rangle + \langle\psi_0|\,I\,|\psi_0\rangle + \langle\psi_0|\,\hat{a}^2\,|\psi_0\rangle]\\ &= \frac{\hbar}{2m\omega}\;[0+2(0)+\langle\psi_0|\psi_0\rangle+0]\\ \Longrightarrow & \boxed{\langle x^2\rangle = \frac{\hbar}{2m\omega}}\;.\end{aligned} \tag{4.10}$$

$$\begin{aligned}\langle p_x^2\rangle &= \langle\psi_0|\,\hat{p}_x^2\,|\psi_0\rangle\\ &= \frac{-\hbar m\omega}{2}\,\langle\psi_0|\,(\hat{a}^{\dagger 2}-2\hat{N}-I+\hat{a}^2)\,|\psi_0\rangle\\ &= \frac{-\hbar m\omega}{2}[\langle\psi_0|\,\hat{a}^{\dagger 2}\,|\psi_0\rangle - 2\,\langle\psi_0|\,\hat{N}\,|\psi_0\rangle - \langle\psi_0|\,I\,|\psi_0\rangle + \langle\psi_0|\,\hat{a}^2\,|\psi_0\rangle]\\ &= \frac{-\hbar m\omega}{2}\;[0-2(0)-\langle\psi_0|\psi_0\rangle+0]\\ \Longrightarrow & \boxed{\langle p_x^2\rangle = \frac{\hbar m\omega}{2}}\;.\end{aligned} \tag{4.11}$$

The variances are thus given by:

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}\;, \tag{4.12}$$

$$\sigma_{p_x}^2 = \langle p_x^2\rangle - \langle p_x\rangle^2 = \frac{\hbar m\omega}{2}\;, \tag{4.13}$$

and hence, we have:

$$\sigma_x\sigma_{p_x} = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2} \geq \frac{\hbar}{2}\;, \tag{4.14}$$

which satisfies the Heisenberg uncertainty relation.

5 Q5

A

$$\begin{aligned}\psi(x,t=0) &= \sqrt{\frac{8}{5a}}\;\left[1+\cos\left(\frac{\pi x}{a}\right)\right]\;\sin\left(\frac{\pi x}{a}\right)\\ &= \sqrt{\frac{8}{5a}}\;\left[\sin\left(\frac{\pi x}{a}\right)+\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right]\\ &= \sqrt{\frac{8}{5a}}\;\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{8}{5a}}\;\frac{1}{2}\;\sin\left(\frac{2\pi x}{a}\right)\\ &= \sqrt{\frac{4}{5}}\;\Phi_1(x)+\sqrt{\frac{1}{5}}\;\Phi_2(x)\;,\end{aligned} \tag{5.1}$$

where $\Phi_n(x)$ is the n-th energy eigenfunction for the infinite square well given by:

$$\Psi_n(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{n\pi x}{a}\right) , \tag{5.2}$$

with corresponding energy eigenfunctions E_n given by:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} . \tag{5.3}$$

B The possible measurements would be states $n = 1$ and $n = 2$:

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} \implies P_{E_1} = |C_1|^2 = \frac{4}{5} \tag{5.4}$$

$$E_2 = \frac{4\pi^2\hbar^2}{2ma^2} \implies P_{E_2} = |C_2|^2 = \frac{1}{5} \tag{5.5}$$

C At any time t, we have:

$$\phi(x,t) = \sqrt{\frac{4}{5}}\ \Phi_1(x)\ e^{-\frac{iE_1t}{\hbar}} + \sqrt{\frac{1}{5}}\ \Phi_2(x)e^{-\frac{iE_2t}{\hbar}} \tag{5.6}$$

D

$$\begin{aligned}\langle E\rangle &= \sum_{n=1}^{2} E_n P_{E_n} \\ &= \frac{4}{5}E_1 + \frac{1}{5}E_2 \\ &= \frac{4}{5}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) + \frac{1}{5}\left(\frac{4\pi^2\hbar^2}{2ma^2}\right) \\ &= \frac{8}{5}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)\end{aligned} \tag{5.7}$$

E Probability of finding the particle in the left half of the well $\implies$ limits of integration from $x = 0$ to $x = a/2$. Hence:

$$\begin{aligned}P &= \int_0^{a/2}\left(\sqrt{\frac{4}{5}}\ \Phi_1^*(x)\ e^{iE_1t/\hbar} + \sqrt{\frac{1}{5}}\ \Phi_2^*(x)\ e^{iE_2t/\hbar}\right)\left(\sqrt{\frac{4}{5}}\ \Phi_1(x)\ e^{-iE_1t/\hbar} + \sqrt{\frac{1}{5}}\ \Phi_2(x)\ e^{-iE_2t/\hbar}\right) dx \\ &= \int_0^{a/2}\left\{\frac{4}{5}\ |\Phi_1|^2 + \frac{2}{5}\ \Phi_1^*\Phi_2\ e^{i(E_1-E_2)t/\hbar} + \frac{2}{5}\ \Phi_2^*\Phi_1\ e^{-i(E_1-E_2)t/\hbar} + \frac{1}{5}\ |\Phi_2|^2\right\} dx .\end{aligned} \tag{5.8}$$

Since Φ_1 and Φ_2 are real, $\implies \Phi_1^* = \Phi_1$, and $\Phi_2^* = \Phi_2$. Thus, $\Phi_1^*\Phi_2 = \Phi_2^*\Phi_1 = \Phi_1\Phi_2$:

$$\begin{aligned} P &= \frac{4}{5}\int_0^{a/2} |\Phi_1|^2 \, dx + \frac{1}{5}\int_0^{a/2} |\Phi_2|^2 \, dx + \ldots \\ &\qquad \ldots \frac{4}{5}\frac{1}{2}\int_0^{a/2} \Phi_1\Phi_2\Big(e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\Big) \, dx \\ &= \frac{4}{5}\int_0^{a/2} |\Phi_1|^2 \, dx + \frac{1}{5}\int_0^{a/2} |\Phi_2|^2 \, dx + \ldots \\ &\qquad \ldots \frac{4}{5}\left[\frac{1}{2}\Big(e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\Big)\right]\int_0^{a/2} \Phi_1\Phi_2 \, dx \\ &= \underbrace{\frac{4}{5}\int_0^{a/2} |\Phi_1|^2 \, dx}_{I_1} + \underbrace{\frac{1}{5}\int_0^{a/2} |\Phi_2|^2 \, dx}_{I_2} + \frac{4}{5}\cos\left[\frac{(E_1-E_2)t}{\hbar}\right]\underbrace{\int_0^{a/2} \Phi_1\Phi_2 \, dx}_{I_3} \ . \end{aligned} \tag{5.9}$$

Let us first evaluate the integral I_3.

$$\begin{aligned} I_3 &= \int_0^{a/2} \Phi_1\Phi_2 \, dx \\ &= \frac{2}{a}\int_0^{a/2} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \, dx \ . \end{aligned} \tag{5.10}$$

We can use the following trigonometric identity: $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ (if you consider yourself a glutton for punishment, feel free to expand the sine terms in terms of complex exponentials. You should obtain the same expression). Letting $\alpha = \frac{3\pi x}{a}$ and $\beta = \frac{\pi x}{a}$, we have:

$$\begin{aligned} I_3 &= \frac{2}{a}\int_0^{a/2} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \, dx \\ &= \frac{2}{a}\int_0^{a/2} \sin\left(\frac{1}{2}\left[\frac{3\pi x}{a} - \frac{\pi x}{a}\right]\right)\sin\left(\frac{1}{2}\left[\frac{3\pi x}{a} + \frac{\pi x}{a}\right]\right) \, dx \\ &= \frac{-1}{2}\frac{2}{a}\int_0^{a/2} \cos\left(\frac{3\pi x}{a}\right) - \cos\left(\frac{\pi x}{a}\right) \, dx \\ &= \frac{-1}{a}\left\{\frac{a}{3\pi}\left[\sin\left(\frac{3\pi x}{a}\right)\right]\Bigg|_0^{a/2} - \frac{a}{\pi}\left[\sin\left(\frac{\pi x}{a}\right)\right]\Bigg|_0^{a/2}\right\} \\ &= \frac{4}{3\pi} \ . \end{aligned} \tag{5.11}$$

Now, we move on and evaluate I_1 and I_2:

$$I_1 = \frac{4}{5}\frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right) \, dx = \frac{4}{10} \ . \tag{5.12}$$

$$I_2 = \frac{1}{5}\frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{2\pi x}{a}\right) \, dx = \frac{1}{10} \ . \tag{5.13}$$

Putting the three integrals together:

$$\begin{aligned} P &= \frac{4}{10} + \frac{1}{10} + \frac{4}{5}\cos\left[\frac{(E_1-E_2)t}{\hbar}\right]\left(\frac{4}{3\pi}\right) \\ &= \frac{1}{2} + \frac{16}{15\pi}\cos\left[\frac{-(E_2-E_1)t}{\hbar}\right] \ . \end{aligned} \tag{5.14}$$

Since $\cos x = \cos -x$:

$$\begin{aligned} P &= \frac{1}{2} + \frac{16}{15\pi} \cos\left[\frac{(E_2 - E_1)t}{\hbar}\right] \\ &= \frac{1}{2} + \frac{16}{15\pi} \cos\left[\frac{3\pi^2\hbar^2}{2ma^2} \frac{t}{\hbar}\right] \\ &= \frac{1}{2} + \frac{16}{15\pi} \cos\left[\frac{3\pi^2\hbar^2 t}{2ma^2}\right] . \end{aligned} \tag{5.15}$$

END OF SOLUTION