NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2015-2016

PH2104 - Analytical Mechanics

Nov/Dec 2015 Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.

2. Answer **ALL** questions.

3. All questions carry **EQUAL** weight.

4. Answer each question beginning on a **FRESH** page of the answer book.

5. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring the **COURSE LECTURE NOTES** for reference.

6. Calculators may be used.

7. You may find the following physical constants useful:

   Newton's gravitational constant, $G = 6.674 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$
   Mass of the Earth, $M_E = 5.972 \times 10^{24}\text{kg}$

Question 1 (Coupled Oscillators)

A model for a diatomic molecule consists of two masses, m and M, connected by a massless spring with spring constant k, as shown below in Figure **1**.

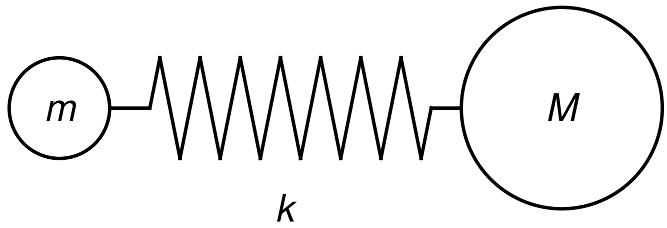

Figure 1: Coupled oscillators.

(a) Find the normal mode (angular) frequencies in terms of m, M, and k.

(12 marks)

(b) Find the *normalized* relative displacements of the masses in each normal mode.

(8 marks)

(c) For each normal mode, show that there is no acceleration of the center of mass.

(5 marks)

Question 2 (Rigid-Body Rotation)

A circular ring of radius R and mass M is suspended by a vertical rod, as shown in Figure **2**.

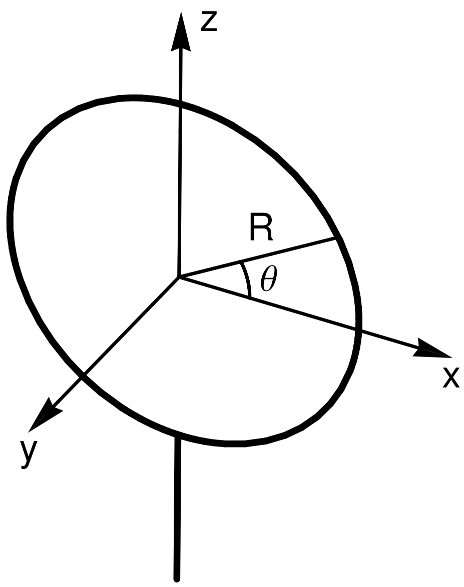

Figure 2: A ring suspended by a vertical rod (the ring is oriented such that it is in a vertical plane).

(a) Write a formula for the mass per unit length, ρ, of the ring (you may assume the ring has uniform density).

(2 marks)

(b) Find the principal moments of inertia of the ring, about its center, in terms of R and M. You may use the following definite integrals:

$$\int_0^{2\pi} \sin^2(\phi)d\phi = \pi \qquad \int_0^{2\pi} \cos^2(\phi)d\phi = \pi$$

(13 marks)

(c) The ring is free to rotate about the vertical axis and does so initially at an angular velocity of 0.500 radians per second. An ant of mass 370 milligrams walks up the vertical rod and walks around the ring. Assuming the ring has a mass of 10.0 grams, calculate the minimum angular velocity of the ring due to the presence of the ant.

(10 marks)

Question 3 (Lagrangian Dynamics)

A triangular shaped mass is free to slide without friction along a horizontal surface. Two other masses are also free to slide without friction on top of this first mass, as illustrated in Figure **3**.

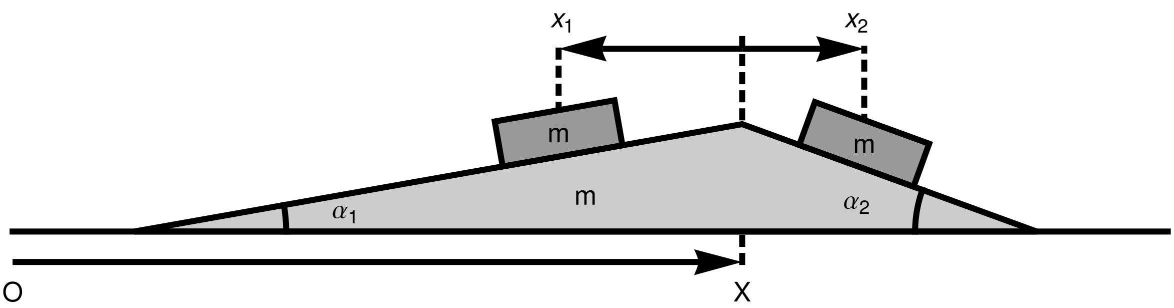

Figure 3: Three sliding masses.

The three masses have equal mass, m, and the triangular shaped mass is characterized by two angles, α_1 and α_2.

(a) Write the kinetic energy, potential energy and Lagrangian for the system. A convenient choice of coordinates is indicated in Figure **3**.

(6 marks)

(b) Use the Euler-Lagrange equations to derive three simultaneous independent equations for the accelerations of the three masses ($\ddot{x}_1$, $\ddot{x}_2$, $\ddot{X}$).

(10 marks)

(c) Prove that the mass that slides on the left-hand slope of the lower mass (i.e., the mass sliding at angle α_1) is accelerated in the upward direction if

$$\tan\alpha_2 > 2\tan\alpha_1 + 3\tan\alpha_1\tan^2\alpha_2.$$

(9 marks)

Question 4 (Central Forces & Planetary Motion)

A space shuttle launches a satellite into a circular orbit of radius r_1 around the center of the Earth. The satellite is to be transferred into a higher circular orbit at radius r_3, by passing through the elliptical orbit characterized by $r_2(\phi)$, as illustrated in Figure **4**.

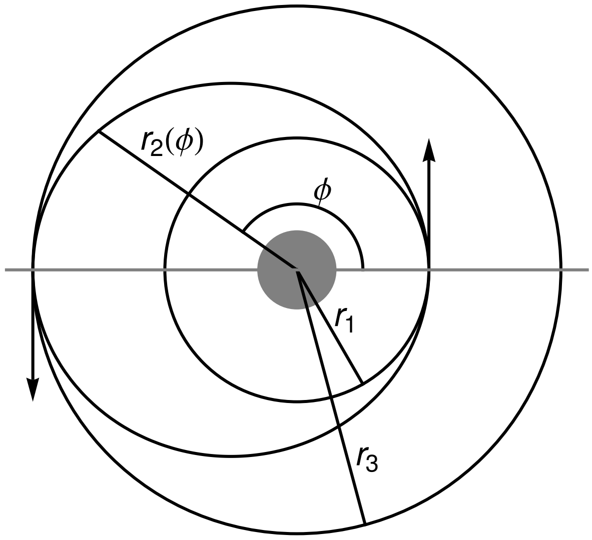

Figure 4: Transfer between two circular orbits. Arrows indicate satellite impulses.

(a) Write an equation describing the geometric shape of the elliptical transfer orbit, $r_2(\phi)$, in terms of the eccentricity, ϵ, and semimajor axis a_2.

(3 marks)

(b) Relate the eccentricity, ϵ, of the transfer orbit to r_1 and r_3.

(6 marks)

(c) It is desired that the final orbit, r_3, is a geostationary orbit such that the position of the satellite is fixed in the sky relative to observers on the Earth. Calculate the required radius of the orbit r_3. You may consider the mass of the satellite as negligible in comparison to that of the Earth. Recall that useful constants are given on the front page of the exam.

(6 marks)

(d) If the initial radius $r_1 = 6,700$km, calculate the eccentricity of the transfer orbit.

(3 marks)

(e) Calculate the time of flight of the satellite between the two circular orbits.

(7 marks)

— End of Paper —

Question 1 Solution

(a)

First write the forces on the three masses, as a function of their displacement:

$$\begin{aligned} F_1 &= k\,(x_2 - x_1) \\ F_2 &= k\,(x_1 - x_2) \end{aligned} \tag{1}$$

According to Newton's second law:

$$\begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let us now substitute for a trial normal mode solution, $\vec{x}(t) = \vec{A}e^{i\omega t}$.

$$-\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Rearranging:

$$\begin{pmatrix} -\frac{k}{m} + \omega^2 & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

The determinant of the above matrix must vanish for a solution, giving:

$$\begin{aligned} \left(-\frac{k}{m} + \omega^2\right)\left(-\frac{k}{M} + \omega^2\right) - \frac{k^2}{mM} &= 0 \\ \omega^4 - \left(\frac{k}{m} + \frac{k}{M}\right)\omega^2 &= 0 \end{aligned}$$

Equality is satisfied for $\omega = 0$ or $\omega^2 = k/m + k/M$.

(b)

The displacements are found by substituting back into the eigenvalue equation.

For $\omega = 0$:

$$0 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

This gives $A_1 = A_2$, corresponding to the normalized eigenvector, where all masses oscillate in-phase:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\omega^2 = k/m + k/M$:

$$-\left(\frac{k}{m} + \frac{k}{M}\right) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

The first line gives:

$$-\left(\frac{k}{m}+\frac{k}{M}\right)A_1 = -\frac{k}{m}A_1 + \frac{k}{m}A_2 \tag{2}$$

or $mA_1 = -MA_2$, corresponding to the normalized eigenvector, where the two masses oscillate out-of phase:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{\sqrt{m^2+M^2}}\begin{pmatrix} M \\ -m \end{pmatrix}$$

(c)

The center of mass position is:

$$R = \frac{mx_1 + Mx_2}{m+M} \tag{3}$$

$$= \frac{mA_1e^{i\omega t} + MA_2A_1e^{i\omega t}}{m+M} \tag{4}$$

For the mode $\omega = 0$, R the time-dependence in the above equation drops out such that R is a constant.

For the mode $\omega^2 = k/m + k/M$:

$$R = \frac{mA_1 + MA_2A_1}{m+M}e^{i\omega t} \tag{5}$$

But the displacements calculated in **(b)** satisfy $mA_1 = -MA_2$ for this mode. Therefore R vanished according to the above equation.

Question 2 Solution

(a)

$$\rho = \frac{M}{2\pi R} \tag{6}$$

(b)

The inertia tensor is defined by a sum over mass elements:

$$I_{ab} = \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + y_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix} \tag{7}$$

Taking the infinitesimal limit, this can be re-written as an integral around the ring by defining $x = R\cos\theta$ and $z = R\sin\theta$, and taking $y = 0$:

$$I_{ab} = \rho \int_0^{2\pi} \begin{pmatrix} R^2\sin^2\theta & 0 & -R^2\sin\theta\cos\theta \\ 0 & R^2\cos^2\theta + R^2\sin^2\theta & 0 \\ -R^2\sin\theta\cos\theta & 0 & R^2\cos^2\theta \end{pmatrix} R d\theta \tag{8}$$

The off-diagonal elements vanish, and using the definite integrals given in the question, we have:

$$I_{ab} = \rho \begin{pmatrix} \pi R^3 & 0 & 0 \\ 0 & 2R^3 & 0 \\ 0 & 0 & \pi R^3 \end{pmatrix} \tag{9}$$

$$= \frac{MR^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

where we used the result from **(a)**. Thus, the principle moments of inertia are $I_{xx} = I_{zz} = MR^2/2$ and $I_{yy} = MR^2$.

(c)

The ant increases the moment of inertia about the vertical axis. The maximum change occurs when the ant has walked a quarter of the way around the ring, giving a new moment of inertia:

$$I'_{zz} = I_{zz} + mR^2 \tag{11}$$

where m is the mass of the ant.

The initial angular momentum of the system is:

$$I_{zz}\omega_0 = \frac{M\omega_0 R^2}{2} \tag{12}$$

where $\omega_0 = 0.5\text{s}^-1$ is the initial angular velocity, and we assume the ant contributes no initial angular momentum when it is still climbing the rod.

The angular momentum when the ant has walked a quarter of the way around the ring is:

$$I'_{zz}\omega = \left(\frac{MR^2}{2} + mR^2\right)\omega \tag{13}$$

where ω is the angular velocity of the ring (and ant) around the vertical axis.

Since angular momentum must be conserved, we now have:

$$I_{zz}\omega_0 = I'_{zz}\omega \tag{14}$$

$$\frac{M\omega_0 R^2}{2} = \left(\frac{MR^2}{2} + mR^2\right)\omega\omega = \frac{\omega_0}{1 + 2\frac{m}{M}} \tag{15}$$

Noting that $m/M = 370 \times 10^{-3}/10.0 = 0.037$, we have $\omega = 0.500/(1 + 2 \times 0.037) \approx 0.466$rad/s.

Question 3 Solution

(a)

To derive the potential energy we note that the vertical coordinates of the masses are related to their horizontal displacements by $y_1 = -x_1 \tan\alpha_1$ and $y_2 = -x_2 \tan\alpha_2$. Therefore:

$$U = mgy_1 + mgy_2 = -mg\left(x_1 \tan\alpha_1 + x_2 \tan\alpha_2\right) \tag{16}$$

The kinetic energy is:

$$T = \frac{1}{2}m\left(\dot{X} - \dot{x}_1\right)^2 + \frac{1}{2}m\left(\dot{X} + \dot{x}_2\right)^2 + \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m\dot{x}_1^2\tan^2\alpha_1 + \frac{1}{2}m\dot{x}_2^2\tan^2\alpha_2 \tag{17}$$

The Lagrangian is:

$$\begin{aligned} L &= T - U & (18)\\ &= \frac{3}{2}m\left(\dot{X}\right)^2 + \frac{1}{2}m\left(1+\tan^2\alpha_1\right)\dot{x}_1^2 + \frac{1}{2}m\left(1+\tan^2\alpha_2\right)\dot{x}_2^2 - m\dot{x}_1\dot{X} + m\dot{x}_2\dot{X} \\ &\qquad + mg\left(x_1\tan\alpha_1 + x_2\tan\alpha_2\right) & (19)\end{aligned}$$

(b)

The Euler-Lagrange Equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} \quad\rightarrow\quad 3m\ddot{X} - m\ddot{x}_1 + m\ddot{x}_2 = 0 \tag{20}$$

$$\rightarrow\quad \ddot{X} = \frac{\ddot{x}_1 - \ddot{x}_2}{3} \tag{21}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \quad\rightarrow\quad m\ddot{x}_1\left(1+\tan^2\alpha_1\right) - m\ddot{X} = mg\tan\alpha_1 \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2} \quad\rightarrow\quad m\ddot{x}_2\left(1+\tan^2\alpha_2\right) + m\ddot{X} = mg\tan\alpha_2 \tag{23}$$

(c)

We need to use the equations derived in **(b)** to find a solution for $\ddot{x}_1$. If $\ddot{x}_1$ is negative, then according to the figure given in the question the first mass will accelerate upwards. To begin with we can use Eq. 21 to eliminate $\ddot{X}$ from Eqs. 22 and 23:

$$\ddot{x}_1\left(1+\tan^2\alpha_1\right) - \frac{\ddot{x}_1 - \ddot{x}_2}{3} = g\tan\alpha_1 \tag{24}$$

$$\ddot{x}_2\left(1+\tan^2\alpha_2\right) + \frac{\ddot{x}_1 - \ddot{x}_2}{3} = g\tan\alpha_2 \tag{25}$$

The first of the above equations can be re-arranged for $\ddot{x}_2$:

$$\ddot{x}_2 = 3g\tan\alpha_1 - \ddot{x}_1\left(2 + 3\tan^2\alpha_1\right) \tag{26}$$

Simplifying Eq. 25 we have:

$$\ddot{x}_2 \left(\frac{2}{3} + \tan^2 \alpha_2 \right) + \frac{\ddot{x}_1}{3} = g \tan \alpha_2 \tag{27}$$

We can use Eq. 25 to eliminate $\ddot{x}_2$ from this equation:

$$\left(3g \tan \alpha_1 - \ddot{x}_1 \left(2 + 3\tan^2 \alpha_1\right)\right) \left(\frac{2}{3} + \tan^2 \alpha_2 \right) + \frac{\ddot{x}_1}{3} = g \tan \alpha_2 \tag{28}$$

Re-arranging for $\ddot{x}_1$:

$$-\ddot{x}_1 \left(1 + 2\tan^2 \alpha_2 + 2\tan^2 \alpha_1 + 3\tan^2 \alpha_1 \tan^2 \alpha_2\right) = g \tan \alpha_2 - g \tan \alpha_1 \left(2 + 3\tan^2 \alpha_2\right) \tag{29}$$

$$\ddot{x}_1 = \frac{-g \left(\tan \alpha_2 - 2\tan \alpha_1 - 3\tan \alpha_1 \tan^2 \alpha_2\right)}{\left(1 + 2\tan^2 \alpha_2 + 2\tan^2 \alpha_1 + 3\tan^2 \alpha_1 \tan^2 \alpha_2\right)} \tag{30}$$

The denominator is always positive. To obtain $\ddot{x}$ negative, we therefore need the numerator to be negative, which indeed requires:

$$\tan \alpha_2 > 2\tan \alpha_1 + 3\tan \alpha_1 \tan^2 \alpha_2 \tag{31}$$

Question 4 Solution

(a)

$$r_2(\phi) = \frac{a_2\left(1-\epsilon^2\right)}{1-\epsilon\cos\phi} \tag{32}$$

(b)

At the beginning of the transfer, $\phi = 0$ and:

$$r_2(0) = \frac{a_2\left(1-\epsilon^2\right)}{1-\epsilon} = r_1 \tag{33}$$

At the end of the transfer, $\phi = \pi$ and:

$$r_2(\pi) = \frac{a_2\left(1-\epsilon^2\right)}{1+\epsilon} = r_3 \tag{34}$$

Putting these two equations together, we obtain:

$$a_2\left(1-\epsilon^2\right) = r_1\left(1-\epsilon\right) = r_3\left(1+\epsilon\right) \tag{35}$$

$$\rightarrow \qquad \epsilon = \frac{r_1 - r_3}{r_1 + r_3} \tag{36}$$

(c)

For a geostationary orbit we need the period to be $\tau = 24\times 60\times 60 = 86400$ seconds.

We also know that the period of an orbit is:

$$\tau = 2\pi\sqrt{\frac{r_3^3}{GM}} \tag{37}$$

Re-arranging for r_3:

$$r_3 = \left(\frac{GM\tau^2}{(2\pi)^2}\right)^{1/3} = 42,240\text{km} \tag{38}$$

where we used the constants given on the front of the exam paper

(d)

Using our equation from **(b)** and our result from **(d)**:

$$\epsilon = \frac{6,700 - 42,240\times 10^4}{6,700 + 42,240\times 10^4} = -0.7262 \tag{39}$$

Note that the sign is irrelevant and only enters through our definition of the angle ϕ.

(e)

The time of flight is given by $\tau_2/2$, that is, one half the period of the elliptical transfer orbit. Given that $\tau = 2\pi\sqrt{a^3/(GM)}$, our real task is to calculate the parameter a. This can be done using the relation considered previously in **(b)**:

$$\frac{a_2\left(1-\epsilon^2\right)}{1-\epsilon} = r_1 \tag{40}$$

$$\rightarrow \qquad a = r_1\frac{1-\epsilon}{(1-\epsilon^2)} = 24,470\text{km} \tag{41}$$

So:

$$\frac{\tau}{2} = \pi\sqrt{\frac{a^3}{GM}} = 19,048\text{s} \tag{42}$$

$$\approx 5.29\text{hours} \tag{43}$$